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ON CONE FUNCTIONS

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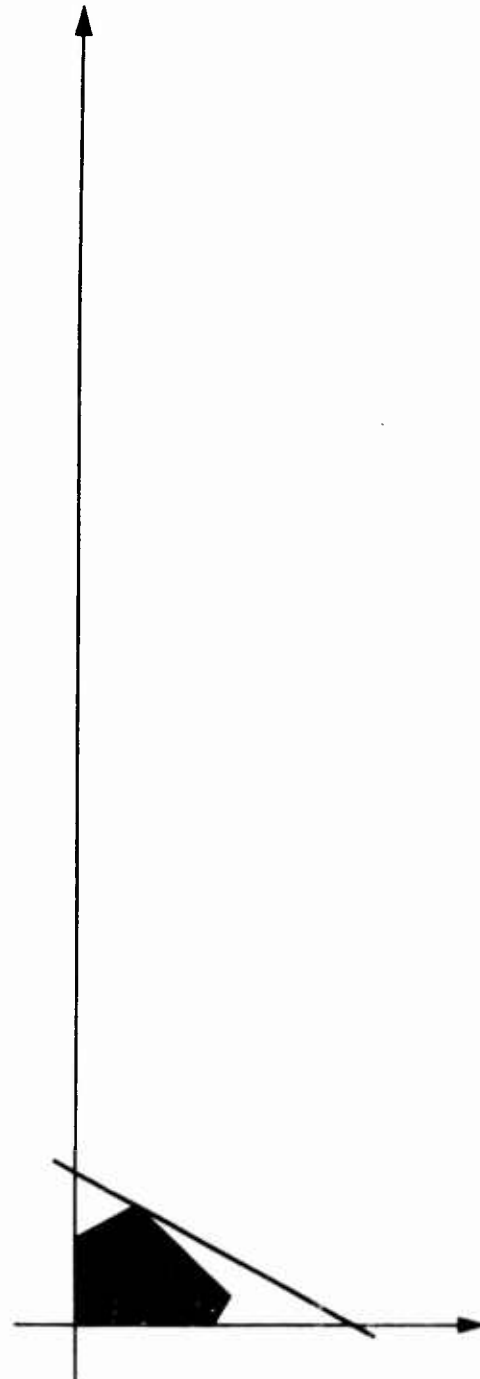
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ON CONE FUNCTIONS

I. INTRODUCTION

In what follows we shall be concerned with generalizations of the following "feasibility" theorem of Fan, Glicksberg, and Hoffman:

THEOREM 1: Let K be a convex subset of R^n, let $f_i : K \to R$, $i = 1, \ldots, m$, be convex functions, then one, and only one, of the following conditions holds:

(1)
    There is an x in K such that

    $f_i(x) < 0$ for all $i = 1, \ldots, m$.

(2)
    There exists a $y = (y_1, \ldots, y_m) \in R^m$ such that

$$y \neq 0, \ y_i \geq 0 \ \text{ all } i, \text{ and } \sum_{i=1}^{m} y_i f_i(x) \geq 0 \ \text{ for all } \ x \in K.$$

The concept underlying our discussion is that of "convexity with respect to a fixed cone C in R^m" of a function $F : K \to R^m$ (K being a convex subset of R^n). This definition turns out to be a natural extension of the case where each component of F is convex in the usual sense, the last being essentially the assumption of Theorem 1, one can then generalize Theorem 1 and its variants to a system of "cone" inequalities (Theorems 3 and 4).

The results discussed here can be shown to hold in a more general framework than that imposed here, e.g., R^n and R^m may be replaced by normed linear spaces satisfying appropriate conditions. However, to be specific, we limit our discussion to the more restricted situation.

II.   BASIC DEFINITIONS

For each positive integer k we denote by R^k the set of all real k - tuples $x = (x_1, \ldots, x_k)$, we sometimes write R in place of R'. If K is a subset of R^k we say K is <u>convex</u> providing $\lambda x + (1 - \lambda)x'$ is in K whenever x and x' are both in K, λ is in R, and $0 \leq \lambda \leq 1$. If C is a nonempty subset of R^k we say that C is a <u>(convex)</u> <u>cone</u> providing C is convex and λx is in C whenever x is in C and λ is non-negative real number.   Henceforth, we shall use "cone" and "convex cone" inter- changeably.   Equivalently, C is a cone providing $\lambda x + \mu x'$ is in C when- ever x and x' are in C and λ and μ are both non-negative real numbers. If x and z are in R^k, $x = (x_1, \ldots, x_k)$, $z = (z_1, \ldots, z_k)$ we write xz^T for the inner product of x and z, i. e.,

$$xz^T = \sum_{i=1}^{k} x_i z_i \quad ;$$

we say that $x \leq z$ providing $x_i \leq z_i$ for all $i = 1, \ldots, k$.   For any subset S of R^k we define the <u>polar</u> of S to be:

$$S^* = R^k \cap \left\{ z \,\middle|\, zx^T \leq 0 \quad \text{for all } x \in S \right\} \quad .$$

It is clear that S^* is always a closed, convex cone.

Whenever we use topological concepts such as "open set", "closed set", etc., it will always be with respect to the usual norm, i.e., $\|x\| = (xx^T)^{1/2}$.   We shall use the following fundamental separation theorem, the proof of which may be found in the literature (cf. [1] or [4]).

THEOREM 2: Let K be a convex subset of R^n, x_0 a point in R^n but x_0 not contained in K, then there must exist $a \in R^n$, $\alpha \in R$, such that $a \neq 0$ and

(3)
$$ax_0 \geq \alpha$$

$$ax \leq \alpha \quad \text{for all} \quad x \in K .$$

If, in addition, K is closed, then we may assume that $ax_0 > \alpha$.

III. CONE FUNCTIONS

The fact that the conclusion of Theorem 1 holds does not depend, in the last analysis, so much on the properties of each of the functions $f_1, \ldots, f_m$ individually but rather on the way f_i are inter-related, that is the relevant properties are those of the vector valued function $F(x) = (f_1(x), \ldots, f_m(x))$. Specifically, to say that each f_i is convex on K is equivalent to saying that:

(4)
for each $x, z \in K$, $\lambda \in R$, $0 \leq \lambda \leq 1$,

$$G(x, z, \lambda) \leq 0$$

where

(5) $\qquad g(x, z, \lambda) = F(\lambda x + (1-\lambda)z) - \lambda F(x) - (1-\lambda)F(z)$.

One observes that (4) simply states that $G(x, z, \lambda)$ is in R_+^m, the non-negative orthant of R^m whenever $x, z \in K$ and $\lambda \in [0, 1]$. It is clear then that a very natural generalization of (4) is the requirement that $G(x, z, \lambda)$ always be in some fixed cone C in R^m. For functions F satisfying that condition one would expect an analog of Theorem 1 to hold with "$f_i(x) < 0$ all i", replaced by "$F(x) \in$ Interior (C)" and "$y_i \geq 0$"

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replaced by "$y \in C^*$". This turns out to be the case and is formalized in Theorem 3. Accordingly, we have:

DEFINITION: Let K be a convex subset of R^n. Let C be a cone in R^m, $F: K \to R^m$, we say that F is a C-function providing $G(x, z, \lambda) \in C$ whenever $x, z \in K$, $\lambda \in [0, 1]$ and G is as in (5).

Let us illustrate the preceding definition. For $m = 1$, the only cones C in R^m are the origin, closed rays from the origin, and the whole line. In the first case a C-function is a linear function; in the second case, a C-function is convex or concave according to whether C is the negative or positive ray; in the last case, any function $F: K \to R^m$ is a C-function. In general, if $C = R^m$ then every function is a C-function, however Theorem 3 is then of no interest because $F(x)$ is in the interior of C for any x in R^n.

For $m = 2$, the class of C-functions does not provide a great amount of new information either, primarily because the closed cones in R^2 are rather simple, they are all finite cones (i.e., the sum of a finite number of rays).

In general, whenever C is a finite cone in R^m, that is there exists an $m \times k$ matrix A such that $C = R^m \cap \{y \mid yA \leq 0\}$, C-functions may be characterized quite simply as follows: let $a^1, \ldots, a^k$ be the column vectors of A then $F: K \to R^m$ is a C-function if, and only if, each of the functions $F(x)a^1, \ldots, F(x)a^k$ is convex on K. Thus, if C is a finite cone, then questions concerning C-functions can be formulated in terms of a finite collection of convex functions, which may, in turn, be accomplished by reference to Theorem 1 or variants of it.

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In case C is not a finite cone (and thus $m \geq 3$) the property that F is a C-function can no longer be stated in terms of convexity of each of a finite collection of functions. In fact, a way of looking at Theorem 3 in case C is not a finite cone is that it represents a generalization of Theorem 1 to a system with infinitely many inequalities.

IV. OPEN FEASIBILITY

The analog of Theorem 1 for C-functions is:

THEOREM 3: Suppose C is a cone in R^m with nonempty interior. Let K be a convex subset of R^n and $F: K \rightarrow R^m$ be a C-function. We conclude that one, and only one, of the following statements holds:

(6)

There is an x such that

$x \in K$ and $F(x) \in$ Interior (C)

(7)

There exists a $y \in C^*$ such that $y \neq 0$ and

$F(x)y^T \geq 0$ for all $x \in K$

Before proving Theorem 3 we require two simple preliminary results:

LEMMA 1: Suppose C is a convex cone in R^m, $y \in$ Interior (C), $z \in C^*$, $z \neq 0$. Conclusion: $yz^T < 0$.

PROOF: Suppose y, z are as above and $yz^T \geq 0$. Since $y \in$ Interior (C), there exists a $\delta > 0$ such that $w \in C$ whenever $\|w - y\| \leq \delta$, let w be any such vector, then $v = 2y - w$ is also in C because $\|v - y\| = \|y - w\| \leq \delta$. Now $y = \frac{1}{2}(v + w)$, $v, w \in C$ and $z \in C^*$, thus $0 \leq yz^T = \frac{1}{2}(vz^T + wz^T) \leq 0$ and $wz^T = 0$. We have just demonstrated that $wz^T = 0$ for all w in some neighborhood of y, contradicting $z \neq 0$.

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LEMMA 2: Let C be a convex cone in R^m, $y \in$ Interior (C), $z \in C$, $\lambda \in R$, $\lambda > 0$. Conclusion: $y + z \in$ Interior (C) and $\lambda y \in$ Interior (C).

PROOF: Since y is in the interior of C we know there exists a $\delta > 0$ such that $u \in C$ whenever $\|u - y\| \leq \delta$; however if $\|(y + z) - w\| \leq \delta$ then $\|y - (w - z)\| \leq \delta$ and $w - z \in C$, but then $w = (w - z) + z \in C$ because C is a cone. Thus $y + z$ is in the interior of C. For the statement: $\lambda y \in$ Interior (C) we have the following sequence of implications: $\|\lambda y - w\| \leq \lambda \delta \implies \|y - \lambda^{-1} w\| < \delta \implies \lambda^{-1} w \in C \implies w \in C$, and thus λy is in the interior of C.

PROOF OF THEOREM 3: The proof that (6) and (7) cannot hold simultaneously follows from Lemma 1 because if (6) and (7) both hold then $y \in C^*$, $x \in K$, $F(x) \in$ Interior (C) then from Lemma 1 we have $F(x) y^T < 0$, contradicting (7). To show that either (6) or (7) holds, let us assume that (6) is false and consider the set:

$$Y = \left\{ y \mid \text{there exist } x \in K \text{ and } \overline{y} \in \text{Int}(C) \text{ with } y = \overline{y} - F(x) \right\}.$$

The fact that (6) is false is equivalent to saying that $y = 0$ is not a member of Y. We intend to show that Y is convex, then apply Theorem 2 to Y, knowing it does not contain the origin, and thus obtain a y satisfying (7). Suppose we have $y_1, y_2 \in Y$, i.e., there exist $\overline{y}_1, \overline{y}_2 \in \text{Int}(C)$, and $x_1, x_2 \in K$ with $y_k = \overline{y}_k - F(x_2)$, $k = 1, 2$, now if $\lambda \in [0, 1)$ we wish to show $y = \lambda y_1 + (1 - \lambda) y_2 \in Y$. Let $u = \lambda \overline{y}_1 + (1 - \lambda) \overline{y}_2$, $v = F(\lambda x_1 + (1-\lambda) x_2) - \lambda F(x_1) - (1-\lambda) F(x_2)$ then $y = u + v - F(\lambda x_1 + (1-\lambda) x_2)$. However, $\overline{y}_1 \in$ Interior (C) and $(1 - \lambda) \overline{y}_2 \in C$ thus, by Lemma 2, $u \in \text{Int}(C)$. Also, because F is a C-function, $v \in C$ and thus by Lemma 1 we have $u + v \in$ Interior (C), showing that Y is convex.

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Next, we know that Y is convex and $y_0 = 0$ is not an element of Y. Applying Theorem 2 we know that there exists an $a \in R^m$, $a \neq 0$, $\alpha \in R$ such that:

(8)
$$0 \geq \alpha$$
$$a[\bar{y} - F(x)]^T \leq \alpha \quad \text{all } \bar{y} \in \text{Int}(C), \ x \in K .$$

From Lemma 2 we know that $\lambda \bar{y} \in \text{Interior}(C)$ whenever $\lambda > 0$ and $y \in \text{Interior}(C)$, thus we may infer from (8) that:

(9)
$$\bar{y}a^T \leq 0 \quad \text{all } \bar{y} \in \text{Int}(C) .$$

Now, by assumption there is a $\bar{y}_0 \in \text{Interior}(C)$, so that if $y \in C$, $\lambda \geq 0$ then by Lemma 2 $\lambda y + y_0 \in \text{Interior}(C)$. We see then, using (9), that

$$\lambda y a^T + \bar{y}_0 a^T \leq 0 \quad \text{all } \lambda \geq 0, \ y \in C .$$

Thus, $ya^T \leq 0$ whenever $y \in C$ and consequently $\underline{a \in C^*}$. Furthermore, for each $\lambda > 0$ we have by Lemma 2 $\lambda \bar{y}_0 \in \text{Interior}(C)$, thus from (8) we have

$$\lambda \bar{y}_0 a^T - F(x)a^T \leq \alpha \leq 0 \quad \text{all } \lambda > 0, \quad x \in K$$

whence it follows that $\underline{F(x)a^T \geq 0}$ for all $x \in K$ and of course $a \neq 0$, thus $y = a$ satisfies all the conditions in (7).

As an immediate corollary of Theorem 3 one obtains Theorem 1 by letting C be the nonpositive orthant and $F(x) = (f_1(x), \ldots, f_m(x))$.

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V. CLOSED FEASIBILITY

The statement of Theorem 1, and similarly of Theorem 3, is in many respects inadequate. Frequently, one encounters situations where it is required, within the framework of Theorem 1 as an example, to find an $x \in K$ with $f_i(x) \leq 0$ for all i, rather than the strict inequalities of (1). It would seem reasonable, given that K is closed, to replace (1) by weak inequalities and correspondingly replace (2) by a statement of the form: "there is a $y = (y_1, \ldots, y_n) \in R^m$ such that $y \geq 0$ and $\sum_{i=1}^{m} y_i f_i(x) > 0$ for all $x \in K$," and expect a true statement. This turns out to be indeed the case when K is a linear variety and all the f_i are linear. In general, with the f_i being convex, it may happen that there is no $x \in K$ such that $f_i(x) \leq 0$ for all i, and yet there is no $y \in R_+^m$ with $\sum_i y_i f_i(x) > 0$ for all $x \in K$, this is illustrated by: $m = 2$, $n = 2$, $K = R_+^2$, $f_1(x_1, x_2) = x_2 - 1$, $f_2(x_1, x_2) = 1 - x_1 x_2 (x_1 + x_2)^{-1} (f_2(0) = 0)$. It is readily checked that f_1 and f_2 are convex; furthermore if $f_1(x) \leq 0$ and x in K then either $x_1 x_2 (x_1 + x_2)^{-1} < x_2 \leq 1$ or $x_2 = 0$, in each case we have $f_2(x) > 0$. However, if there exist $y_1, y_2 \geq 0$ such that $y_1 f_1(x) + y_2 f_2(x) > 0$ for all x in K, i.e.,

$$y_1 (x_2 - 1) + y_2 \left[1 - \frac{x_1 x_2}{x_1 + x_2} \right] > 0 \quad \text{all } x_1, x_2 \geq 0$$

then, letting $x_2 = 0$, $x_1 = 1$, we get $y_2 > y_1$. On the other hand, letting x_1 become arbitrarily large, we must have: $y_1(x_2 - 1) + y_2(1 - x_2) > 0$ for all $x_2 \geq 0$, and thus $y_1 = y_2$, a contradiction.

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It thus follows, in particular, that the rephrasing of Theorem 3 in terms of weak inequalities, i.e., replacing "Interior (C)" in (6) and "$F(x)y^T \geq 0$" by "$F(x)y^T > 0$" in (7), need not always hold. We can show, however, that under certain regularity assumptions on C, K and F the statement in question does hold:

THEOREM 4: Let C, K and F be as in Theorem 3, let

$$H = \left\{ y \mid \text{there exist } x \text{ in } k \text{ and } \overline{y} \text{ in } C \text{ with } y = \overline{y} - F(x) \right\},$$

then H is a convex set. Furthermore, if H is also closed and the statement:

(10)
$$\text{There is an } x \text{ such that}$$
$$x \text{ is in } K \text{ and } F(x) \text{ is in } C$$

is false, then the statement:

(11)
$$\text{There is a } y \in C^* \text{ such that}$$
$$F(x)y^T > 0 \text{ for all } x \text{ in } K$$

is a true statement.

PROOF: As in the proof of Theorem 3, the convexity of H is a direct consequence of the assumptions on C, K and F. The fact that (10) is false is equivalent to saying that $y_0 = 0$ is not in H, thus if H is <u>closed</u> and convex then, from Theorem 2, it follows that there must exist an $a \in R^m$ and $\alpha \in R$ such that:

(12)
$$ay^T \leq \alpha < ay_0^T = 0, \quad \text{all } y \in H$$

i.e.,

(13)
$$a\overline{y}^T - F(x)a^T \leq \alpha < 0, \quad \text{all } \overline{y} \in C, \ x \in K.$$

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However, $0 \in C$ and also $\lambda y \in C$ whenever $y \in C$ and $\lambda \geq 0$ and thus we infer from (13) that:

(14)
$$ay^T \leq 0 , \qquad \qquad \text{all} \quad y \in C$$
$$- F(x)a^T \leq \alpha < 0, \qquad \text{all} \quad x \in K$$

and (14) states that the vector a is precisely the one required for (11).

Q.E.D.

NOTE: We have stated Theorem 4 in slightly different form than Theorem 3, however, it is quite obvious that (10) and (11) cannot be simultaneously true.

It is worth noting that the condition that H be closed though certainly sufficient in order that (11) hold in case (10) is false, is by no means necessary. For one thing, as is clear from the proof of Theorem 4, we can actually get a positive lower bound for $F(x)y^T$ on K (namely the number $-\alpha$) when H is closed; however, consider the case $m = n = 1$, $K = R_+$, $C = R_-$ and $F(x) = (1+x)^{-1}$ then F is a C-function (because F is convex) but $x \in K$, $F(x) \in C$ has no solution because $F(x) > 0$ for all $x \in K$. Now $C^* = R_+$ and in fact any $y \in C^*$ will satisfy (11). Nevertheless, no matter what $y \in C^*$ we take $F(x)y^T$ has no positive lower bound as x ranges over K (by letting x become arbitrarily large we can make $F(x)$ arbitrarily close to zero). Thus, in this case, H is not closed. Other situations where the closedness of H is not necessary arise when K itself is not closed.

In a future note it is intended to relate to cone functions the following characterization of differentiable convex functions: Suppose K is an open convex set in R^n, $f : K \to R$ has all second partial derivatives then f is convex if, and only if, the quadratic form $[f_{ij}(x)]$ is positive-semi-definite for each x in K.

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